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Gauge theories with graded differential Lie algebras

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Abstract

We present a mathematical framework of gauge theories that is based upon a skew-adjoint Lie algebra and a generalized Dirac operator, both acting on a Hilbert space.

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1 Introduction

This paper precises the author's previous article[1], in which we proposed a mathematical calculus towards gauge field theories based upon graded differential Lie algebras. Given a skew-adjoint Lie algebra $\mathfrak{g}$, a representation π of $\mathfrak{g}$ on a Hilbert space h_0 as well as an [unbounded] operator D and a grading operator Γ on h_0, we developed a scheme providing connection and curvature forms to build physical actions. The general part of our exposition was on a very formal level, we worked with unbounded operators (even the splitting of a bounded in two unbounded operators) without specification of the domain.

In the present paper, we correct this shortcoming. The idea is to introduce a second Hilbert space h_1, which is the domain of the unbounded operator D. Now, D is a linear continuous operator from h_1 to h_0, and the just mentioned splitting involves continuous operators only. Moreover, the awkward connection theory in the previous paper is resolved in a strict algebraic description in terms of normalizers of graded Lie algebras. Finally, our construction of the universal graded differential Lie algebra is considerably simplified (thanks to a hint by K. Schmüdgen).

The scope of our framework is the construction of Yang–Mills–Higgs models in noncommutative geometry[2]. The standard procedure[3,4] starts from spectral triples with real structure[5,6] and is limited to the standard model[7]. The hope is[8] that the replacement of the unital associative $*$-algebra in the prior Connes–Lott prescription[9] by a skew-adjoint Lie algebra admits representations general enough to construct grand unified theories. For a realization of this strategy see refs. [10,11,12]. We discuss the relation to the axiomatic formulation[6] of noncommutative geometry in the last section.

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2 The algebraic setting

Let $\mathfrak{g}$ be a skew-adjoint Lie algebra, $a^* = -a$ for all $a \in \mathfrak{g}$. Let h_0, h_1 be Hilbert spaces, where h_1 is dense in h_0. Denoting by $\mathcal{B}(h_0)$ and $\mathcal{B}(h_1)$ the algebras of linear continuous operators on h_0 and h_1, respectively, we define $\mathcal{B} := \mathcal{B}(h_0) \cap \mathcal{B}(h_1)$. The vector space of linear continuous mappings from h_1 to h_0 is denoted by $\mathcal{L}$. Let π be a representation of $\mathfrak{g}$ in $\mathcal{B}$. Let $D \in \mathcal{L}$ be a generalized Dirac operator with respect to $\pi(\mathfrak{g})$. This means that D has an extension to a selfadjoint operator on h_0, that $[D, \pi(a)] \in \mathcal{L}$ even belongs to $\mathcal{B}$ for any $a \in \mathfrak{g}$ and that the resolvent of D is compact. Finally, let $\Gamma \in \mathcal{B}$ be a grading operator, i.e. Γ^2 is the identity on both h_0 and h_1, $[\Gamma, \pi(a)] = 0$ on both h_0, h_1 for any $a \in \mathfrak{g}$ and $D\Gamma + \Gamma D = 0$ on h_1 extends to 0 on h_0. This setting was called *L-cycle* in ref. [1], referring to a *Lie*-algebraic version of a *K-cycle*, the former name[2,9] for spectral triple[5,6].

The standard example of this setting $(\mathfrak{g}, h_0, h_1, D, \pi, \Gamma)$ is

$$\begin{aligned} \mathfrak{g} &= C^\infty(X) \otimes \mathfrak{a}\,, & h_0 &= L^2(\mathcal{S}) \otimes \mathbb{C}^F\,, \\ h_1 &= W_1^2(\mathcal{S}) \otimes \mathbb{C}^F\,, & D &= \mathrm{i}\partial\!\!\!/ \otimes 1_F + \boldsymbol{\gamma} \otimes \mathcal{M}\,, \\ \pi &= \mathrm{id} \otimes \hat{\pi}\,, & \Gamma &= \boldsymbol{\gamma} \otimes \hat{\Gamma}\,. \end{aligned} \tag{1}$$

Here, $C^\infty(X)$ denotes the algebra of real-valued smooth functions on a compact Riemannian spin manifold X, $\mathfrak{a}$ is a skew-adjoint matrix Lie algebra, $L^2(\mathcal{S})$ denotes the Hilbert space of square integrable sections of the spinor bundle $\mathcal{S}$ over X, $W_1^2(\mathcal{S})$ denotes the Sobolev space of square integrable sections of $\mathcal{S}$ with generalized first derivative, $\mathrm{i}\partial\!\!\!/$ is the Dirac operator of the spin connection, $\boldsymbol{\gamma}$ is the grading operator on $L^2(\mathcal{S})$ anti-commuting with $\mathrm{i}\partial\!\!\!/$, $\hat{\pi}$ is a representation of $\mathfrak{a}$ in $\mathrm{M}_F\mathbb{C}$ and $\hat{\Gamma}$ a grading operator on $\mathrm{M}_F\mathbb{C}$ commuting with $\hat{\pi}(\mathfrak{a})$ and anti-commuting with $\mathcal{M} \in \mathrm{M}_F\mathbb{C}$.

3 The universal graded differential Lie algebra Ω

For $\mathfrak{g}$ being a real Lie algebra we consider the real vector space $\mathfrak{g}^2 = \mathfrak{g} \times \mathfrak{g}$, with the linear operations given by $\lambda_1(a_1, a_2) + \lambda_2(a_3, a_4) = (\lambda_1 a_1 + \lambda_2 a_3, \lambda_1 a_2 + \lambda_2 a_4)$, for $a_i \in \mathfrak{g}$ and $\lambda_i \in \mathbb{R}$. Let T be the tensor algebra of $\mathfrak{g}^2$, equipped with the $\mathbb{N}$–grading structure $\deg((a,0)) = 0$ and $\deg((0,a)) = 1$, and linear extension to higher degrees, $\deg(t_1 \otimes t_2) = \deg(t_1) + \deg(t_2)$, for $t_i \in T$. Defining $T^n = \{t \in T : \deg(t) = n\}$, we have $T = \bigoplus_{n\in\mathbb{N}} T^n$ and $T^k \otimes T^l \subset T^{k+l}$. We regard T as a graded Lie algebra with graded commutator given by $[t^k, t^l] := t^k \otimes t^l - (-1)^{kl}\, t^l \otimes t^k$, for $t^n \in T^n$.

Let $\tilde{\Omega} = \bigoplus_{n\in\mathbb{N}} \tilde{\Omega}^n = \sum [\mathfrak{g}^2, [\dots[\mathfrak{g}^2, \mathfrak{g}^2]\dots]]$ be the $\mathbb{N}$-graded Lie subalgebra of T [due to the graded Jacobi identity] given by the set of sums of repeated commutators of elements of $\mathfrak{g}^2$. Let I' be the vector subspace of $\tilde{\Omega}$ of sums of elements of the following type:

$$[(a,0),(b,0)] - ([a,b],0)\ , \qquad [(a,0),(0,b)] + [(0,a),(b,0)] - (0,[a,b])\ , \tag{2}$$

for $a, b \in \mathfrak{g}$. The first part extends the Lie algebra structure of $\mathfrak{g}$ to the first component of $\mathfrak{g}^2$ and the second part plays the rôle of a Leibniz rule, see below.

Obviously, $I := I' + [\mathfrak{g}^2, I'] + [\mathfrak{g}^2, [\mathfrak{g}^2, I']] + \dots$ is an $\mathbb{N}$-graded ideal of $\tilde{\Omega}$, so that $\Omega := \bigoplus_{n\in\mathbb{N}} \Omega^n$, $\Omega^n := \tilde{\Omega}^n/(I \cap \Omega^n)$ is an $\mathbb{N}$-graded Lie algebra.

On T we define recursively a graded differential as an $\mathbb{R}$-linear map $d : T^n \to T^{n+1}$ by

$$\begin{aligned} d(a,0) &= (0,a) \;, & d(0,a) &= 0 \;, \\ d((a,0)\otimes t) &= d(a,0)\otimes t + (a,0)\otimes dt \;, & d((0,a)\otimes t) &= -(0,a)\otimes dt \;, \end{aligned} \tag{3}$$

for $a \in \mathfrak{g}$ and $t \in T$. One easily verifies $d^2 = 0$ on T and the graded Leibniz rule $d(t^k \otimes t^l) = dt^k \otimes t^l + (-1)^k t^k \otimes dt^l$, for $t^n \in T^n$. Thus, d defined by (3) is a graded differential of the tensor algebra T and of the graded Lie algebra T as well, $d[t^k, t^l] = [dt^k, t^l] + (-1)^k [t^k, dt^l]$.

Due to $d\mathfrak{g}^2 \subset \mathfrak{g}^2$ we conclude that d is also a graded differential of the graded Lie subalgebra $\tilde{\Omega} \subset T$. Moreover, one easily checks $dI' \subset I'$, giving $dI \subset I$. Therefore, $(\Omega\,, [\;,\;]\,, d)$ is a graded differential Lie algebra, with the graded differential d given by $d(\varpi + I) := d\varpi + I$, for $\varpi \in \tilde{\Omega}$.

We extend the involution $* : a \mapsto -a$ on $\mathfrak{g}$ to an involution of T by $(a,0)^* = -(a,0)$, $(0,a)^* = -(0,a)$ and $(t_1 \otimes t_2)^* = t_2^* \otimes t_1^*$, giving $[t^k, t^l]^* = -(-1)^{kl}[t^{k*}, t^{l*}]$. Clearly, this involution extends to Ω. The identity $a = -a^*$ yields $\omega^{k*} = -(-1)^{k(k-1)/2}\omega^k$, for any $\omega^k \in \Omega^k$.

The graded differential Lie algebra Ω is universal in the following sense:

Proposition 1 *Let $\Lambda = \bigoplus_{n\in\mathbb{N}} \Lambda^n$ be an $\mathbb{N}$-graded Lie algebra with graded differential* d *such that*

i) $\Lambda^0 = \pi(\mathfrak{g})$ *for a surjective homomorphism π of Lie algebras,*

ii) Λ *is generated by $\pi(\mathfrak{g})$ and $\mathrm{d}\pi(\mathfrak{g})$ as the set of repeated commutators.*

Then there exists a differential ideal $I_\Lambda \subset \Omega$ fulfilling $\Lambda \cong \Omega/I_\Lambda$.

Proof. Define a linear surjective mapping $p : \Omega \to \Lambda$ by

$$p((a,0)) = \pi(a) \;, \qquad p(d\omega) = \mathrm{d}(p(\omega)) \;, \qquad p([\omega, \tilde{\omega}]) := [p(\omega), p(\tilde{\omega})] \;,$$

for $a \in \mathfrak{g}$ and $\omega, \tilde{\omega} \in \Omega$. Because of $d \ker p \subset \ker p$, $I_\Lambda = \ker p$ is the desired differential ideal of Ω. $\square$

4 The graded differential Lie algebra Ω_D

Using the grading operator Γ we define on $\mathcal{L}$ and $\mathcal{B}$ a $\mathbb{Z}_2$–grading structure, the even subspaces carry the subscript 0 and the odd subspaces the subscript 1. Then, the graded commutator $[\,.\,,.\,]_g : \mathcal{L}_i \times \mathcal{B}_j \to \mathcal{L}_{(i+j) \bmod 2}$ is defined by

$$[A_i, B_j]_g := A_i \circ B_j - (-1)^{ij} B_j \circ A_i \equiv -(-1)^{ij}[B_j, A_i]_g \;, \tag{4}$$

where $B_j \in \mathcal{B}_j$ and $A_i \in \mathcal{L}_i$. If $A_i \in \mathcal{B}_i$ then $[\,.\,,.\,]_g$ maps h_1 to h_1 and h_0 to h_0.

Using the elements π and D of our setting we define a linear mapping $\pi : \Omega \to \mathcal{B}$ by

$$\begin{aligned} \pi((a,0)) &:= \pi(a) \;, & \pi((0,a)) &:= [-\mathrm{i}D, \pi(a)]_g \;, \\ \pi([\omega^k, \omega^l]) &:= [\pi(\omega^k), \pi(\omega^l)]_g \;, \end{aligned} \tag{5}$$

for $a \in \mathfrak{g}$ and $\omega^n \in \Omega^n$. The selfadjointness of D on h_0 implies that π is involutive, $\pi(\omega^*) = (\pi(\omega))^*$.

Note that $\pi(\Omega)$ is not a graded *differential* Lie algebra. The standard procedure to construct such an object is to define $\mathcal{J} = \ker \pi + d \ker \pi \subset \Omega$. It is easy to show that $\mathcal{J}$ is a graded differential ideal of Ω, providing the graded differential Lie algebra

$$\Omega_D = \bigoplus\nolimits_{n\in\mathbb{N}} \Omega_D^n \;, \qquad\qquad \Omega_D^n := \Omega^n/\mathcal{J}^n \cong \pi(\Omega^n)/\pi(\mathcal{J}^n) \;, \tag{6}$$

where $\mathcal{J}^n = \mathcal{J} \cap \Omega^n$. One has $\Omega_D^0 \cong \pi(\Omega^0) \equiv \pi(\mathfrak{g})$ and $\Omega_D^1 \cong \pi(\Omega^1)$. By construction, the differential d on Ω_D is given by $d(\pi(\omega^n) + \pi(\mathcal{J}^n)) := \pi(d\omega^n) + \pi(\mathcal{J}^{n+1})$, for $\omega^n \in \Omega^n$.

It is very useful to consider an extension of the second formula of (5), $\pi(d(a,0)) := [-\mathrm{i}D, \pi((a,0))]_g$, to higher degrees:

$$\pi(d\omega^n) = [-\mathrm{i}D, \pi(\omega^n)]_g + \sigma(\omega^n) \;, \quad \omega^n \in \Omega^n \;. \tag{7}$$

It turns out[1] that $\sigma : \Omega \to \mathcal{L}$ is a linear mapping recursively given by

$$\begin{aligned} \sigma((a,0)) = 0 \;, \qquad\qquad & \sigma((0,a)) = [D,[D,\pi(a)]_g]_g \;, \\ \sigma([\omega^k,\omega^l]) = [\sigma(\omega^k), \pi(\omega^l)]_g &+ (-1)^k [\pi(\omega^k), \sigma(\omega^l)]_g \;. \end{aligned} \tag{8}$$

Equation (7) has an important consequence: Putting $\omega^n \in \ker \pi$ we get

$$\pi(\mathcal{J}^{n+1}) = \{\sigma(\omega^n) \;:\; \omega^n \in \Omega^n \cap \ker \pi\} \;. \tag{9}$$

The point is that $\sigma(\Omega)$ can be computed from the last equation (8) once $\sigma(\Omega^1)$ is known. Then one can compute $\pi(\mathcal{J})$ and obtains with (7) an explicit formula for the differential on Ω_D.

5 Connections

We define the graded normalizer $N_{\mathcal{L}}(\pi(\Omega))$ of $\pi(\Omega)$ in $\mathcal{L}$, its vector subspace $\mathcal{H}$ compatible with $\pi(\mathcal{J})$ and the graded centralizer $\mathcal{C}$ of $\pi(\Omega)$ in $\mathcal{L}$ by

$$\begin{aligned} N^k_{\mathcal{L}}(\pi(\Omega)) &= \{\eta^k \in \mathcal{L}_{k \bmod 2} \;:\; \eta^k \text{ has} \left\{\begin{smallmatrix}\text{selfadjoint}\\ \text{skew-adjoint}\end{smallmatrix}\right\} \text{extension for } \tfrac{k(k-1)}{2} \left\{\begin{smallmatrix}\text{odd}\\ \text{even}\end{smallmatrix}\right\}, \\ &\qquad [\eta^k, \pi(\omega^n)]_g \in \pi(\Omega^{k+n}) \;\; \forall \omega^n \in \Omega^n \,,\; \forall n \in \mathbb{N}\,\} \,, \\ \mathcal{H}^k &= \{\eta^k \in N^k_{\mathcal{L}}(\pi(\Omega)) \;:\; [\eta^k, \pi(j^n)]_g \in \pi(\mathcal{J}^{k+n}) \;\; \forall j^n \in \mathcal{J}^n\,\} \,, \\ \mathcal{C}^k &= \{c^k \in N^k_{\mathcal{L}}(\pi(\Omega)) \;:\; [c^k, \pi(\omega)]_g = 0 \;\; \forall \omega \in \Omega\,\} \,. \end{aligned} \tag{10}$$

Here, the linear continuous operator $[\eta^k, \pi(\omega^n)]_g : h_1 \to h_0$ must have its image even in the subspace $h_1 \subset h_0$ and must have an extension to a linear continuous operator on h_0. For each degree n we have the following system of inclusions:

$$\begin{array}{ccccccc} \mathcal{L} & \supset & \mathcal{H}^n & \supset & \pi(\Omega^n) & \supset & \pi(\mathcal{J}^n) \\ & & \cup & & \cap & & \\ & & \mathcal{C}^n & & \mathcal{B} & \subset & \mathcal{L} \end{array} \tag{11}$$

The graded Jacobi identity and Leibniz rule define the structure of a graded differential Lie algebra on $\hat{\mathcal{H}} = \bigoplus_{n\in\mathbb{N}} \hat{\mathcal{H}}^n$, with $\hat{\mathcal{H}}^n = \mathcal{H}^n/(\mathcal{C}^n+\pi(\mathcal{J}^n))$:

$$\begin{aligned}
&[[\eta^k+\mathcal{C}^k+\pi(\mathcal{J}^k)\,,\,\eta^l+\mathcal{C}^l+\pi(\mathcal{J}^l)]_g\,,\,\pi(\omega^n)+\pi(\mathcal{J}^n)]_g \\
&\qquad := [\eta^k,[\eta^l,\pi(\omega^n)]_g]_g - (-1)^{kl}[\eta^l,[\eta^k,\pi(\omega^n)]_g]_g + \pi(\mathcal{J}^{k+l+n})\,, \\
&[d(\eta^k+\mathcal{C}^k+\pi(\mathcal{J}^k))\,,\,\pi(\omega^n)+\pi(\mathcal{J}^n)]_g \\
&\qquad := \pi\circ d\circ\pi^{-1}([\eta^k,\pi(\omega^n)]_g) - (-1)^k[\eta^k,\pi(d\omega^n)]_g + \pi(\mathcal{J}^{k+n+1})\,,
\end{aligned} \tag{12}$$

for $\eta^n \in \mathcal{H}^n$ and $\omega^n \in \Omega^n$.

The lesson is that $\pi(\Omega)$ and its ideal $\pi(\mathcal{J})$ give rise not only to the graded differential Lie algebra Ω_D but also to $\hat{\mathcal{H}}$, both being natural. It turns out that it is the differential Lie algebra $\hat{\mathcal{H}}$ which occurs in our connection theory:

Definition 2 *Within our setting, a connection ∇ and its associated covariant derivative $\mathcal{D}$ are defined by*

i) $\mathcal{D} \in \mathcal{L}_1$ *with selfadjoint extension,*
ii) $\nabla : \Omega_D^n \to \Omega_D^{n+1}$ *is linear,*
iii) $\nabla(\pi(\omega^n)+\pi(\mathcal{J}^n)) = [-\mathrm{i}\mathcal{D},\pi(\omega^n)]_g + \sigma(\omega^n) + \pi(\mathcal{J}^{n+1})\;, \quad \omega^n \in \Omega^n.$

The operator $\nabla^2 : \Omega_D^n \to \Omega_D^{n+2}$ is called the curvature of the connection.

This definition states that the covariant derivative $\mathcal{D}$ generalizes the operator D of the setting and the connection ∇ generalizes the differential d. In particular, both $\mathcal{D}$ and ∇ are related via the same equation iii) as D and d are according to (7).

Proposition 3 *Any connection/covariant derivative has the form $\nabla = d + [\rho+\mathcal{C}^1\,,\;.\;]_g$ and $\mathcal{D} = D + \mathrm{i}\rho$, for $\rho \in \mathcal{H}^1$. The curvature is $\nabla^2 = [\theta,\;.\;]$, with $\theta = d\hat{\rho} + \frac{1}{2}[\hat{\rho},\hat{\rho}]_g \in \hat{\mathcal{H}}^2$, where $\hat{\rho} = \rho + \mathcal{C}^1 \in \hat{\mathcal{H}}^1$.*

Proof. There is a canonical pair of connection/covariant derivative given by $\nabla = d$ and $\mathcal{D} = D$. If $(\nabla^{(1)},\mathcal{D}^{(1)})$ and $(\nabla^{(2)},\mathcal{D}^{(2)})$ are two pairs of connections/covariant derivatives, we get from iii)

$$(\nabla^{(1)}-\nabla^{(2)})(\pi(\omega^n)+\pi(\mathcal{J}^n)) = [\nabla_h^{(1)}-\nabla_h^{(2)},\pi(\omega^n)]_g + \pi(\mathcal{J}^{n+1}).$$

This means that $\rho := \nabla_h^{(1)}-\nabla_h^{(2)} \in \mathcal{H}^1$ is a concrete representative and $\nabla^{(1)}-\nabla^{(2)} = [\hat{\rho},\,.\,]_g$, where $\hat{\rho} = \rho + \mathcal{C}^1 \in \hat{\mathcal{H}}^1$. The formula for θ is a direct consequence of (12). □

6 Gauge transformations

The exponential mapping defines a unitary group

$$\begin{aligned}
\mathcal{U} := \{\textstyle\prod_{\alpha=1}^N \exp(v_\alpha) : \quad & \exp(v_\alpha) := \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{k=1}^\infty \frac{1}{k!}(v_\alpha)^k\,, \\
& v_\alpha \in \mathcal{H}^0 \cap \mathcal{B}\,,\; dv_\alpha - [-\mathrm{i}D, v_\alpha] \in \mathcal{C}^1\;\}\,.
\end{aligned} \tag{13}$$

Due to $\exp(v)A\exp(-v) = A + \sum_{k=1}^\infty \frac{1}{k!}([v,[v,\dots,[v,A]\dots]])_k$, where $(\quad)_k$ contains k commutators of $A \in \mathcal{L}$ with v, we have a natural degree-preserving representation Ad of $\mathcal{U}$ on $\mathcal{H}$, $\mathrm{Ad}_u(\eta^n) = u\eta^n u^* \in \mathcal{H}^n$, for $\eta^n \in \mathcal{H}^n$ and $u \in \mathcal{U}$.

Definition 4 *In our setting, the gauge group is the group $\mathcal{U}$ defined in* (13). *Gauge transformations of the connection and the covariant derivative are given by*

$$\nabla \mapsto \nabla' := \mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*} \;, \quad \mathcal{D} \mapsto \mathcal{D}' := u\mathcal{D}u^* \;, \quad u \in \mathcal{U}.$$

Note that the consistency relation iii) in Definition 2 reduces on the infinitesimal level to the condition $dv_\alpha - [-\mathrm{i}D, v_\alpha] \in \mathcal{C}^1$ in (13). The gauge transformation of the curvature form reads $\theta \mapsto \theta' = \mathrm{Ad}_u \theta$.

7 Physical action

We borrow the integration calculus introduced by Connes to noncommutative geometry[2,5] and summarize the main results. Let E_n be the eigenvalues of the compact operator [compactness was assumed in the setting] $|D|^{-1} = (DD^*)^{-1/2}$ on h_0, arranged in decreasing order. Here, the finite dimensional kernel of D is not relevant so that $E_1 < \infty$. The K-cycle (h_0, D) over the C^*-algebra $\mathcal{B}(h_0)$ is called d^+-summable if $\sum_{n=1}^N E_n = O(\sum_{n=1}^N n^{-1/\mathrm{d}})$. Equivalently, the partial sum of the first N eigenvalues of $|D|^{-\mathrm{d}}$ has [at most] a logarithmic divergence as $N \to \infty$ so that $|D|^{-\mathrm{d}}$ belongs to the [two-sided] Dixmier trace class ideal $\mathcal{L}^{(1,\infty)}(h_0)$. Therefore, $f\,|D|^{-\mathrm{d}} \in \mathcal{L}^{(1,\infty)}(h_0)$ for any $f \in \mathcal{B}(h_0)$, and the Dixmier trace provides for $f > 0$ a linear functional $f \mapsto \mathrm{Tr}_\omega(f\,|D|^{-\mathrm{d}}) = \mathrm{Lim}_\omega \frac{1}{\ln N} \sum_{n=1}^N \mu_n \in \mathbb{R}^+$. Here, μ_n are the eigenvalues of $f\,|D|^{-\mathrm{d}}$ and the limit Lim_ω involves an appropriate limiting procedure ω. The Dixmier trace fulfills $\mathrm{Tr}_\omega(f\,|D|^{-\mathrm{d}}) = \mathrm{Tr}_\omega(ufu^*\,|D|^{-\mathrm{d}})$, for unitary $u \in \mathcal{B}(h_0)$.

Let $\theta_0^* : h_0 \to h_1$ be the uniquely determined adjoint of a representative $\theta_0 : h_1 \to h_0$ of the curvature form $\theta \in \hat{\mathcal{H}}^2$. It follows $\theta_0\theta_0^* \in \mathcal{B}(h_0)$ so that we propose the following definition for the physical action:

Definition 5 *The bosonic action S_B and the fermionic action S_F of the connection ∇ and covariant derivative $\mathcal{D}$ are given by*

$$\begin{aligned} S_B(\nabla) &:= \min_{\mathrm{j}^2 \in \mathcal{C}^2 + \pi(\mathcal{J}^2)} \mathrm{Tr}_\omega((\theta_0 + \mathrm{j}^2)(\theta_0 + \mathrm{j}^2)^* |D|^{-\mathrm{d}}) \;, \\ S_F(\psi, \mathcal{D}) &:= \langle \psi, \mathcal{D}\psi \rangle_{h_0} \;, \quad \psi \in h_1 \;, \end{aligned} \tag{14}$$

where $\langle\ ,\ \rangle_{h_0}$ is the scalar product on h_0.

The bosonic action S_B is independent of the choice of the representative θ_0. Thus, we can take the canonical dependence of the gauge potential ρ,

$$\theta_0 = \{-\mathrm{i}D, \rho\} + \tfrac{1}{2}\{\rho, \rho\} + \sigma \circ \pi^{-1}(\rho) \;,$$

where $\sigma \circ \pi^{-1}$ is supposed to be extended from $\pi(\Omega^1)$ to $\mathcal{H}^1$. It is unique up to elements of $\mathcal{C}^2 + \pi(\mathcal{J}^2)$. Since the Dixmier trace is positive, the element $\mathrm{j}_0^2 \in \mathcal{C}^2 + \pi(\mathcal{J}^2)$ at which the minimum in (14) is attained is the unique solution of the equation

$$\mathrm{Tr}_\omega((\theta_0 + \mathrm{j}_0^2)(\mathrm{j}^2)^* |D|^{-\mathrm{d}}) = 0 \;, \quad \forall\, \mathrm{j}^2 \in \mathcal{C}^2 + \pi(\mathcal{J}^2) \;.$$

It is clear that the action (14) is invariant under gauge transformations

$$\nabla \mapsto \mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*} \;, \quad \mathcal{D} \to u\mathcal{D}u^* \;, \quad \psi \mapsto u\psi \;, \quad u \in \mathcal{U} \;. \tag{15}$$

Note that our gauge group as defined in (13) is always connected, which means that we have no access to 'big' gauge transformations. Note further that there exist Lie groups having the same Lie algebra. In that case there will exist fermion multiplets ψ which can be regarded as multiplets of different Lie groups. For the bosonic sector only the Lie algebra is important, so one can have the pathological situation of a model with identical particle contents and identical interactions, but different gauge groups. We consider such gauge theories as identical.

8 Remarks on the standard example

Recall (5) that the general form of an element $\tau^1 \in \pi(\Omega^1)$ is

$$\tau^1 = \sum_{\alpha, z \geq 0} [\pi(a^z_\alpha), [\dots[\pi(a^1_\alpha), [-\mathrm{i}D, \pi(a^0_\alpha)]]\dots]] \;, \qquad a^i_\alpha \in \mathfrak{g} \;. \tag{16}$$

For $a^i_\alpha = f^i_\alpha \otimes \hat{a}^i_\alpha \in C^\infty(X) \otimes \mathfrak{a}$ we get with (1)

$$\begin{aligned} \tau^1 = \sum_{\alpha, z \geq 0} \Big(& f^z_\alpha \cdots f^1_\alpha \not{\partial}(f^0_\alpha) \otimes \hat{\pi}([\hat{a}^z_\alpha, [\dots[\hat{a}^1_\alpha, \hat{a}^0_\alpha]\dots]]) \\ & + f^z_\alpha \cdots f^1_\alpha f^0_\alpha \boldsymbol{\gamma} \otimes [\hat{\pi}(\hat{a}^z_\alpha), [\dots[\hat{\pi}(\hat{a}^1_\alpha), [-\mathrm{i}\mathcal{M}, \hat{\pi}(\hat{a}^0_\alpha)]]\dots]] \Big) \;. \end{aligned} \tag{17}$$

Let us first assume that $\mathfrak{a}$ is semisimple. In this case the two lines in (17) are independent. The first line belongs to $\Lambda^1 \otimes \hat{\pi}(\mathfrak{a})$, because the gamma matrices occurring in $\not{\partial}$ provide a 1-form basis. In physical terminology, these Lie algebra-valued 1-forms are Yang–Mills fields acting via the representation $\mathrm{id} \otimes \hat{\pi}$ on the fermions. In the second line of (17) we split $\mathcal{M}$ into generators of irreducible representations of $\mathfrak{a}$, tensorized by generation matrices. Obviously, these irreducible representations are spanned after taking the commutators with $\hat{\pi}(\hat{a}^i_\alpha)$. Thus, the second line of (17) contains sums of function-valued representations of the matrix Lie algebra [times $\boldsymbol{\gamma}$ and generation matrices], which are physically interpreted as Higgs fields. In other words, the prototype τ^1 of a connection form (=gauge potential) describes representations of both Yang–Mills and Higgs fields on the fermionic Hilbert space.

From a physical point of view, this is a more satisfactory picture than the usual noncommutative geometrical construction of Yang–Mills–Higgs models[3,4]. Namely, descending from Connes' noncommutative geometry[2,5,6] there is only a limited set of Higgs multiplets possible[13]: Admissible Higgs multiplets are tensor products $\mathbf{n} \otimes \mathbf{m}^*$ of fundamental representations (and their complex conjugate) $\mathbf{n}, \mathbf{m}$ of simple gauge groups, where the adjoint representation never occurs. This rules out[7] the construction of interesting physical models. In our framework there are no such restrictions and – depending on the choice of $\mathcal{M}$ and h – Higgs fields in any representation of a Lie group are possible. Thus, a much larger class of physical models can be constructed.

The treatment of Abelian factors $\mathfrak{a}'' \subset \mathfrak{a}$ in our approach is somewhat tricky. One remarks that in the first line of (17) only the (z=0)-component of $\mathfrak{a}''$ survives. The consequence is that linear independence of the two lines in (17) is not automatical. Thus, to avoid pathologies, we need a condition[1] between $\mathcal{M}$ and the representations of $\mathfrak{a}$ to assure independence. The u(1)-part of the standard model is admissible in this sense.

The second consequence of the missing (z>0)-components in the first line is that the spacetime 1-form part of Abelian factors in τ^1 is a total differential $\not{\partial}(f_0^0) \in d\Lambda^0 \subset \Lambda^1$. This seems to be a disaster at first sight for the description of Abelian Yang–Mills fields. However, our gauge potential lives in the bigger space $\mathcal{H}^1 \supset \pi(\Omega^1)$. Always if there is a part $d\Lambda^0 \otimes \pi(\mathfrak{a}'')$ in $\pi(\Omega^1)$ there is a part $\Lambda^1 \otimes \pi(\mathfrak{a}'')$ in $\mathcal{H}^1$. There can be even further contributions from $\mathcal{H}^1$ to the gauge potential, which are difficult to control in general. Fortunately, it turns out[1] that after imposing a locality condition for the connection (which is equivalent to saying that ρ commutes with functions), possible additional $\mathcal{H}^1$-degrees of freedom are either of Yang–Mills type or Higgs type.

This framework of gauge field theories was successfully applied to formulate the standard model[10], the flipped SU(5) × U(1)-grand unification[11] and SO(10)-grand unification[12]. It is not possible to describe pure electrodynamics. The reason is that in the Abelian case the curvature form $\theta \neq 0$ commutes with all elements of $\pi(\Omega)$. Hence, it belongs to the graded centralizer $\mathcal{C}^2$ and is projected away in the bosonic action (14).

9 Do the axioms of noncommutative geometry extend to the Lie algebraic setting?

The present status of noncommutative geometry is that this theory is governed by seven axioms[6]. In the commutative case, these axioms provide the algebraic description of classical spin manifolds. The question now is whether or not our Lie algebraic version, which is in close analogy with the prior Connes–Lott formulation[9] of noncommutative geometry, can also be brought into contact with Connes' axioms. We list and discuss below the axioms in their form they would have in terms of Lie algebras.

1) *Dimension: $|D|^{-1}$ is an infinitesimal of order $\frac{1}{\mathrm{d}}$, i.e. the eigenvalues E_n of $|D|^{-1}$ grow as $n^{-1/\mathrm{d}}$, where* d *is an even natural number.*

3) *Smoothness: For any $a \in \mathfrak{g}$, both a and $[D, a]$ belong to the domain of δ^m, where $\delta(\,.\,) := [|D|,\,.\,]$.*
   The axioms 1) and 3) can be directly transferred to the Lie algebraic setting. We cannot treat the odd-dimensional case as the grading operator Γ is essential to detect the sign for the graded commutator.

4) *Orientability:* Connes requires the $\mathbb{Z}_2$-grading operator Γ to be the image under π of a Hochschild d-cycle. We are not going to touch the extension of Hochschild homology to Lie algebras, but even a requirement such as $\Gamma \in \pi(\Omega^{\mathrm{d}})$ is problematic. For the standard example we have the decomposition $\Gamma = \boldsymbol{\gamma} \otimes \hat{\Gamma}$, and the comparison with the general form[1] of $\pi(\Omega^{\mathrm{d}})$ yields that

$\hat{\Gamma}$ has to be the image under $\hat{\pi}$ of the non-abelian part of $\mathfrak{a}$. In all models we have studied so far this is not the case. It seems to be impossible to maintain orientability in our framework. The grading operator Γ, which commutes with $\pi(\mathfrak{g})$ and anti-commutes with D, is an extra piece which has no relation with orientability.

7) *Reality: There exists an anti-linear isometry* $J : h_i \to h_i$ *such that* $[\pi(a), J\pi(b)J^{-1}] = 0$ *for all* $a, b \in \mathfrak{g}$, $J^2 = \epsilon$, $JD = DJ$ *and* $J\Gamma = \epsilon'\Gamma J$, *with* $\epsilon = (-1)^{\mathrm{d(d+2)/8}}$ *and* $\epsilon' = (-1)^{\mathrm{d}/2}$.

2) *First order:* $[[D, \pi(a)], J\pi(b)J^{-1}] = 0$ *for all* $a, b \in \mathfrak{g}$.
Both axioms 7) and 2) can be trivially fulfilled as soon as an anti-linear involution $\mathcal{I}$ on h_i is available. It suffices to define

$$J := \begin{pmatrix} 0 & \epsilon\mathcal{I}^{-1} \\ \mathcal{I} & 0 \end{pmatrix}, \qquad h_i \mapsto \begin{pmatrix} h_i \\ h_i \end{pmatrix}, \qquad D \mapsto \begin{pmatrix} D & 0 \\ 0 & \mathcal{I}D\mathcal{I}^{-1} \end{pmatrix},$$
$$\pi(a) \mapsto \begin{pmatrix} \pi(a) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Gamma \mapsto \begin{pmatrix} \Gamma & 0 \\ 0 & \epsilon'\mathcal{I}\Gamma\mathcal{I}^{-1} \end{pmatrix}.$$

The question is whether there are nontrivial real structures which also satisfy the other axioms. The existence of the real structure J (Tomita's involution) is a central piece of Connes' theory. It has proved very useful in understanding the commuting electroweak and strong sectors of the standard model. The same idea could be applied to our formulation of the standard model[10]. For other gauge theories[11,12], however, a nontrivial real structure J seems to be rather disturbing as it requires the fermions to sit in (generalized) adjoint representations. To achieve this one had to add auxiliary u(1)-factors, which is in contradiction to the grand unification philosophy.

5) *Finiteness and absolute continuity:* Connes requires $h_\infty = \bigcap_m \mathrm{domain}(D^m)$ to be a finite projective module. Thus, our task would be to define the notion of a finite projective module over a Lie algebra $\mathfrak{g}$ and the Lie analogues of the K-groups. We are not aware of these structures, but without them it is impossible to talk about generalizations of the index pairing of D with the K-groups and of

6) *Poincaré duality.*

In conclusion, our Lie algebraic version of noncommutative geometry is not a possible generalization of classical spin manifolds, or at least there is a lot to do to derive the Lie analogues of standard algebraic structures. Our approach provides a powerful tool to build gauge field theories with spontaneous symmetry breaking, the price for this achievement is the lost of any contact with spin manifolds.

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